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Quantum Mechanics on Topologically Nontrivial Spaces

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The formulation of quantum mechanics on topologically nontrivial spaces is discussed. It is pointed out that the "obstacles" are represented by cohomology groups and not only by $\pi_1(M)$ as usually stated. Some widespread errors and misunderstandings are cleared up.

Recently much attention has been devoted to formulation of quantum mechanics on topologically nontrivial spaces [see, e.g., Schulman (1981) and Polonyi (1988)]. The subject is interesting in itself and it is also believed that it has important physical implications: topologically nontrivial configurations play an important role in solid state physics, the confinement mechanism, multidimensional field theory, and so on. The Aharonov–Bohm effect is the best known influence of nontrivial topology on physical systems (Aharonov and Bohm, 1959; Ryder, 1985). In this effect and also in the so-called Hosotani mechanism (Hosotani, 1989; Green *et al.*, 1986; Sładkowski, 1990*a,b*; Mańka *et al.*, 1989) we have a nontrivial background gauge field with vanishing curvature. These effects are caused by nontrivial holonomy groups (Kobayashi and Nomizu, 1963). Due to the vanishing of the curvature, we must have a non-simply-connected configuration space to have a nontrivial holonomy group; see (Kobayashi and Nomizu, 1963) for details.

Misled by this, one usually supposes that the configuration space must not be simply connected to reveal topological effects (Schulman, 1981; Polonyi, 1988). The point of this note is that in general, the responsible "topological defects" are described by cohomology groups.

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To show this, let us analyze the quantum mechanics of a particle interacting with a (classical) electromagnetic field [$U(1)$-gauge field]. The stress-energy tensor $F_{\mu\nu}$ might be closed but not exact:

$$dF = 0 \qquad \text{and} \qquad F \neq dA \tag{1}$$

This may happen if the second cohomology group of the configuration space M is nontrivial, $H^2(M, \mathbb{R}) \neq 0$ (dim $M > 1$). Equation (1) means that the gauge potential exists only locally and the principle of minimal coupling

$$p^{\mu} \to p^{\mu} - \frac{e}{c} A^{\mu} \tag{2}$$

cannot be applied.

To deal with such situations the "multivalued theory" has been developed (Polonyi, 1988; Dubrovin *et al.*, 1982; Novikov, 1982). One often even claims that physical quantities are only well defined locally (cf. Polonyi, 1988, and references therein). This seems strange. A free particle has a well-defined momentum p^{μ}; why should this also not be so after "turning on" a magnetic field (the momentum, however, may not be describable in one coordinate patch, but this does not cause any conceptual problem)? We can hardly accept such "locality," especially when it is not necessary.

The problem is caused by an oversimplified description. The $U(1)$-gauge potential is a connection form ω (Kobayashi and Nomizu, 1963; Trautman, 1981) on a principal fiber bundle $P[M, U(1)]$. Physicists prefer the local description defined by a (local) section σ of the bundle P (gauge condition):

$$A_{\mu}(x)\, dx^{\mu} = \sigma^{*}\omega \tag{3}$$

The vector potential $A_{\mu}(x)$ is used in (2). If the appropriate bundle is nontrivial, there is no global section and the vector potential is defined only locally.

But does this mean that the physical quantities are defined only locally? The physical gauge field is described by the connection form ω and is well defined globally (and single valued!). The locality is only an artifact of the formalism being used.

We have solved our conceptual problems. Unfortunately, it is usually a horrible task to try to solve the appropriate equations using the fiber bundle formalism. We are compelled to use the local description to obtain a differential equation that we can try to solve. Usually, we face the problem of multivaluedness (Schulman, 1981; Polonyi, 1988; Ryder, 1985; Novikov, 1982). This can be avoided by reformulating the quantum mechanical problem on the covering space of the configuration. Unfortunately, this is not always so. To be as general as possible, let us suppose that we face a problem where a k-form ω is closed but not exact. We are looking for a minimal covering space $\tilde{M}$,

$$\tilde{M} \xrightarrow{p} M \tag{4}$$

so that $p^*\omega = dA$. Let us choose a basis $\{\gamma_1, \ldots, \gamma_n\}$ in the kth homology group $H_K(M, \mathbb{Z})$ so that

$$\int_{\gamma_i} \omega = \begin{cases} 0 & i > j \\ \delta_i \neq 0, & i \leq j \end{cases} \tag{5}$$

If all the δ_i vanish, then ω is exact. If ω is not exact, we must look for a (universal) covering space. Such a space always exists due to the following theorem (Greenberg, 1967).

Theorem 1. Every connected manifold has a universal covering space that is also a manifold. Unfortunately, the following theorems (Spanier, 1966; Whitehead, 1978) say that only the $k = 1$ case can be successfully dealt with in such a way.

Theorem 2. If $p: (X, x_0) \rightarrow [B, p(x_0)]$ is a covering, then

$$(p_*)_n : \pi_n(X, x_0) \rightarrow \pi_n[B, p(x_0)]$$

is an isomorphism for all $n \geq 2$.

Theorem 3 (Hurewicz). If X is simply connected, then the first non-vanishing homotopy group is isomorphic to the first nonvanishing integer homology group.

The formulation of the problem in a "covering space" with nondiscrete fiber does not help. Mathematicians have introduced the notion of fibration (Spanier, 1966; Whitehead, 1978). We have the following facts.

Theorem 4. Let $p: X \rightarrow B$ be the projection of a fiber bundle, and suppose that B is paracompact. Then p is a fibration.

Theorem 5. Let $p: X \rightarrow B$ be a fibration whose fiber F is contractible in X. Then $\pi_n(B) = \pi_n(X) \oplus \pi_{n-1}(F)$ for all $n \geq 2$.

Theorem 6. If the fibration $p: X \rightarrow B$ has a cross section, then $\pi_n(X) = \pi_n(F) \oplus \pi_n(B)$ for all $n \geq 2$.

Theorem 7. If the fiber F of the fibration $p: X \rightarrow B$ has only constant path, then $(P_*)_n : \pi_n(X, X_0) \rightarrow \pi_n(B, X_0)$ is an isomorphism for all $n \geq 2$.

From Theorems 1-7 it follows that only the case with $k = 1$ can be simplified by describing it in the appropriate covering space. The Aharonov-Bohm effect belongs to that class.

Example 1. Let us suppose that, according to (5), $K=1$, $\delta_i \neq 0$, for $i \geq j$. The monodromy group of the minimal covering space is exactly $\mathbb{Z}^j$, the free Abelian group with j generators. This covering space may not be simply connected (and universal). In the Aharonov–Bohm effect we have $j=1$ and covering is universal.

Most of the technical problems are in fact not caused by nonexactness of the appropriate differential forms, but simply by the lack of global coordinate systems. There is little we can do unless the covering space admits global coordinates. Nevertheless, the description in the universal covering space is usually much easier even if the covering space is not simply connected; e.g., S^2 is much more convenient as a configuration space than $\mathbb{R}P^2$. We must be careful not to impose any physical meaning on "fictitious" multivaluedness. A solution of the problem formulated in the covering space "joins different" points of the fiber and defines a representation of the first homotopy group of the true configuration space on the set values of the wave function. Only the solutions defining the trivial representation represent true solutions. One often forgets about that (Schulman, 1981; Polonyi, 1988; Carlen and Loffredo, 1989).

Example 2. Let $M=S^1$. The universal covering space is $\mathbb{R}$. We can describe a free particle moving on the circle in $\mathbb{R}$ via the Hamilton operator:

$$H=\frac{-1}{2m}\frac{d^2}{d\varphi^2} \tag{6}$$

but only the periodic solutions describe the particle moving on the circle. The twisted solutions

$$\psi(x+2\pi)=e^{i\kappa}\psi(x) \tag{7}$$

represent another problem. Namely, $M=[0, 2\pi]$ with the boundary problems defined by (7). Problems with different κ's are unitary, not equivalent, and describe physically different configurations. There is no need to "generalize" quantum mechanics as done, e.g., in Schulman (1981), Polonyi (1988), and Carlen and Loffredo (1989). We cannot give any physical meaning to the (abuse of) a coordinate system! We simply should find the correct Hilbert space to describe it. But this is another problem.

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